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An intersection result for families of abelian varieties

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Abstract

We show that on a non isotrivial family of abelian varieties over a smooth complete curve effective relatively ample divisors are strictly nef. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Let $\pi : \mathscr{A} \to B$ be a family of abelian varieties over $\mathbf{C}$ of relative dimension g over a smooth projective curve i.e. $\mathscr{A}$ and B are complex smooth projective varieties of dimension $g+1$ and 1, π is a proper smooth morphism such that $A_b = \pi^{-1}(b)$ is an abelian variety for all $b \in B$.

The main result of this paper is that if $\pi : \mathscr{A} \to B$ is not isotrivial then an effective relatively ample divisor must intersect any curve (Corollary 4.5). Moreover, if the general fiber is simple then any two subvarieties, of complementary codimension and not contained in fibers, have a non-zero intersection number (Corollary 4.6). Here, since the base B has dimension 1, general fiber means A_b, for $b \in B$ outside a countable set of B.

We will give two independent proofs of the main result. A basic ingredient of both the proofs is the action of $\mathbf{Z}$ on the rational cohomology and the rational Chow ring of an abelian scheme; see Sections 2 and 3.

The first proof of the main result is in Section 4 where we obtain it as a corollary of:

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Theorem 4.3. *Let $\pi:\mathscr{A}\to B$ be an abelian scheme of dimension g over a smooth projective curve B, with zero section e. Let L be a relatively ample line bundle on $\mathscr{A}$. If $deg(c_1(L)\cdot e(B))<(\frac{1}{4})\, deg\, c_1(\omega_{\mathscr{A}/B})$ then L is not effective.*

The proof of this theorem is based on the Grothendieck–Riemann–Roch Theorem and on a result of Mumford on π_*L.

The second proof of the main result, given in Section 5, has a more differential geometrical flavor. With a local $\mathscr{C}^\infty$ trivialization we show that a relatively ample divisor $\mathscr{X}$ must intersect the n-torsion locus $\mathscr{A}[n]\subset\mathscr{A}$ for infinitely many $n\in\mathbf{N}$. Since the class of $\mathscr{A}[n]$ in the Chow group of $\mathscr{A}$ is a non-zero multiple of the class of the zero section (see Remark 3.2), the main result follows provided that not every irreducible component of $\mathscr{A}[n]$ which meets $\mathscr{X}$ is contained in $\mathscr{X}$. To exclude that possibility we use a theorem of Raynaud on the finiteness of the number of torsion points on subvarieties of abelian varieties and an argument which shows that we may assume the general fiber of $\mathscr{A}$ to be simple.

2. Action of multiplication by integers on the cohomology

2.1. Let $\pi:\mathscr{A}\to B$ be an abelian scheme over $\mathbf{C}$ of relative dimension g over a smooth connected projective variety B of dimension d, with zero section $e:B\to\mathscr{A}$.

2.2. Multiplication by $k:\mathscr{A}\to\mathscr{A}$ $(k\in\mathbf{Z})$ induces maps

$$k^*:H^p(B,R^i\pi_*\mathbf{Q})\to H^p(B,R^i\pi_*\mathbf{Q})\quad\text{and}\quad k^*=k^i,$$

that is, multiplication by the scalar k^i. Combined with the Leray filtration, this gives a canonical decomposition:

$$H^n(\mathscr{A},\mathbf{Q})=\bigoplus_{i=\max\{0,n-2d\}}^{\min\{n,2g\}} H^{n-i}(B,R^i\pi_*\mathbf{Q}).$$

Thus, any $Z\in H^n(\mathscr{A},\mathbf{Q})$ can be written as

$$Z=\sum_{i=\max\{0,n-2d\}}^{\min\{n,2g\}} Z_{n-i}\quad\text{with } k^*Z_{n-i}=k^iZ_{n-i} \tag{2.2.1}$$

and $Z_{n-i}\in H^{n-i}(B,R^i\pi_*\mathbf{Q})$.

2.3. Remark

1. For $n=2d$ and $i=0$ one has $H^{2d}(B,\pi_*\mathbf{Q})\simeq H^{2d}(B,\mathbf{Q})\simeq\mathbf{Q}$ and the class $[A_b]$ of a fiber is a basis (note $k^*A_b=A_b$).
2. For $n=2g+2d$, $H^{2g+2d}(\mathscr{A},\mathbf{Q})=H^{2d}(B,R^{2g}\pi_*\mathbf{Q})\simeq\mathbf{Q}$ and the class of a point is a basis.

3. For $n=2g$, $H^0(B,R^{2g}\pi_*\mathbf{Q})\simeq H^0(B,\mathbf{Q})\simeq\mathbf{Q}$. The class $[e(B)]$ of the zero section is a basis of $H^0(B,R^{2g}\pi_*\mathbf{Q})$, (in fact, since $k_*e(B)=e(B)$, from $k_*k^*=k^{2g}id$ it follows that $k^*e(B)=k^{2g}e(B)$).

4. For the same reason also the class of the m-torsion locus, $[\mathscr{A}[m]]$, lives in $H^0(B,R^{2g}\pi_*\mathbf{Q})$, for any $m\in\mathbf{N}$. So in particular it is a positive multiple of $[e(B)]$.

2.3. Remark. The cup product gives a duality between the spaces $H^n(\mathscr{A},\mathbf{Q})$ and $H^{2g+2d-n}(\mathscr{A},\mathbf{Q})$ with the property

$$(k^*W\cdot k^*Z)=k^*(W\cdot Z)=k^{2g}(W\cdot Z).$$

Thus, if we decompose $W=\sum_i W_i$, $Z=\sum_j Z_j$ as in (2.2.1), we get

$$(W_i\cdot Z_j)=0 \quad \text{if } i+j\neq 2d. \tag{2.4.1}$$

In particular, since $[e(B)]=[e(B)]_0\in H^{2g}(\mathscr{A},\mathbf{Q})$, for any $Z\in H^{2d}(\mathscr{A},\mathbf{Q})$ we have

$$([e(B)]\cdot Z)=([e(B)]\cdot Z_{2d}). \tag{2.4.2}$$

Note that $Z_{2d}\in H^{2d}(B,R^0\pi_*\mathbf{Q})$ is a multiple of $[A_b]$.

3. Action of multiplication by integers on the Chow ring

3.1. Let $\pi:\mathscr{A}\to B$ be an abelian scheme as in 2.1. By Fourier theory on abelian schemes [2, Theorem 2.19], we have the following decomposition of $CH^p(\mathscr{A})_{\mathbf{Q}}=CH^p(\mathscr{A})\otimes\mathbf{Q}$ in eigenspaces for the multiplication by k:

$$CH^p(\mathscr{A})_{\mathbf{Q}}=\bigoplus_{s=\mathrm{Max}(p-g,2p-2g)}^{\mathrm{Min}(2p,p+d)} CH^p(\mathscr{A})_s \quad \text{with } k^*Z_s=k^{2p-s}Z_s$$

for $Z_s\in CH^p(\mathscr{A})_s$. Thus, any cycle can be written as

$$Z=\sum_s Z_s\in CH^p(\mathscr{A})_{\mathbf{Q}} \quad \text{with } Z_s\in CH^p(\mathscr{A})_s.$$

3.2. Remark. The Fourier transform gives an isomorphism (cf. [2, 2.18. v])

$$\mathscr{F}:CH^p(\mathscr{A})_s\to CH^{g-p+s}(\mathscr{A})_s,$$

in particular for $p=g$, $CH^g(\mathscr{A})_0\simeq CH^0(\mathscr{A})_0$. Note that $e(B)\in CH^g(\mathscr{A})_0$ (cf. Remark 2.3.3) and because $CH^0(\mathscr{A})_{\mathbf{Q}}=CH^0(\mathscr{A})_0=\mathbf{Q}\mathscr{A}$, one has

$$CH^g(\mathscr{A})_0=\mathbf{Q}e(B).$$

In particular, this implies that the m-torsion locus $\mathscr{A}[m]$ is a (positive) multiple of $e(B)$ in $CH^g(\mathscr{A})_{\mathbf{Q}}$, generalizing to the level of Chow groups the result in Remark 2.3.4. This property was already observed by Looijenga in [4].

3.3. Remark. The intersection product in the Chow ring satisfies

$$CH^p(\mathscr{A})_s \otimes CH^q(\mathscr{A})_t \to CH^{p+q}(\mathscr{A})_{s+t}.$$

For $p=d$ and $p+q=g+d$ with $d \le g$, we have the decompositions $CH^d(\mathscr{A})_{\mathbf{Q}} = \bigoplus_{s=d-g}^{2d} CH^d(\mathscr{A})_s$ $CH^{g+d}(\mathscr{A})_{\mathbf{Q}} = \bigoplus_{s=2d}^{g+d} CH^{g+2d}(\mathscr{A})_s$. Thus, the intersection product of $e(B) \in CH^g(\mathscr{A})_0$ with any $Z = \sum_s Z_s \in CH^d(\mathscr{A})_{\mathbf{Q}}$ is actually given by

$$(e(B) \cdot Z) = (e(B) \cdot Z_{2d}). \tag{3.3.1}$$

3.4. Lemma. *For an abelian scheme* $\pi : \mathscr{A} \to B$ *as in* 2.1, *with* $\dim B = 1$, *we have*

$$CH^1(\mathscr{A})_{\mathbf{Q}} = CH^1(\mathscr{A})_0 \oplus CH^1(\mathscr{A})_1 \oplus CH^1(\mathscr{A})_2,$$

so $CH^1(\mathscr{A})_s = 0$ *for* $s < 0$. *Moreover,*

$$CH^1(\mathscr{A})_2 = \pi^* Pic(B)_{\mathbf{Q}}.$$

Proof. We use the exact sequence

$$0 \to Pic^0(\mathscr{A}) \to CH^1(\mathscr{A}) \to NS(\mathscr{A}) \to 0$$

which implies that

$$CH^1(\mathscr{A})_{\mathbf{Q}} = Pic^0(\mathscr{A})_{\mathbf{Q}} \oplus NS(\mathscr{A})_{\mathbf{Q}}.$$

Since $NS(\mathscr{A})_{\mathbf{Q}} \subset H^2(\mathscr{A}, \mathbf{Q})$, the Néron–Severi group can be decomposed as follows:

$$NS(\mathscr{A})_{\mathbf{Q}} = NS_0 \oplus NS_1 \oplus NS_2, \qquad NS_i = NS(\mathscr{A})_{\mathbf{Q}} \cap H^{2-i}(B, R^i\pi_* \mathbf{Q}).$$

Hence $NS_s = 0$ for $s < 0$. Using Remark 2.3.1 we find

$$NS_2 = \mathbf{Q}[A_b] = \pi^* H^2(B, \mathbf{Q}). \tag{3.4.1}$$

Now we consider $Pic^0(\mathscr{A})_{\mathbf{Q}}$. Let $\tau : \mathscr{A}^t \to B$ be the dual abelian scheme. Then an element of $\mathscr{A}^t(B)$, the group of sections of τ, is an isomorphism class of line bundles L on $\mathscr{A}$ with $e^* L \cong \mathcal{O}_B$ and whose restriction to each fiber of π is algebraically equivalent to zero (see [3, p. 2]). Therefore, we have

$$Pic^0(\mathscr{A})_{\mathbf{Q}} \subset \mathscr{A}^t(B)_{\mathbf{Q}} \oplus \pi^* Pic^0(B)_{\mathbf{Q}}$$

and this decomposition is stable under k^*. We first show that

$$\mathscr{A}^t(B)_{\mathbf{Q}} = \mathscr{A}^t(B)_0 \oplus \mathscr{A}^t(B)_1 \quad \text{with } k^* Z_s = k^{2-s} Z_s$$

for $Z_s \in \mathscr{A}^t(B)_s$.

Let L be a line bundle on $\mathscr{A}$ which corresponds to an element in $\mathscr{A}^t(B)$ and whose class $c_1(L) \in \mathscr{A}^t(B)_{\mathbf{Q}}$ satisfies $c_1(L)_s = 0$ for $s = 0$, 1. The inclusion of a fiber A_b of π into $\mathscr{A}$ is equivariant for multiplication by integers and $CH^1(A_b)_{\mathbf{Q}} = CH^1(A_b)_0$ $\oplus CH^1(A_b)_1$. Thus, the restriction of $c_1(L)$ to each fiber is trivial. Therefore $L^{\otimes n}$, for some $n>0$, restricts to $\mathcal{O}_{A_b}$ for all $b\in B$. By functoriality of $\mathscr{A}^t$ (in particular, $(\mathscr{A}^t)_b=(A_b)^t=Pic^0(A_b)$), the sections of τ corresponding to $L^{\otimes n}$ and $\mathcal{O}_{\mathscr{A}}$ must then be the same. Therefore $L^{\otimes n}\cong\mathcal{O}_{\mathscr{A}}$ and thus $c_1(L)=0\in\mathscr{A}^t(B)_{\mathbf{Q}}$. We conclude that $\mathscr{A}^t(B)_s=0$ if $s\neq 0$, 1.

Since $\pi\circ k=\pi$ we get $k^*\pi^*=\pi^*$, hence

$$\pi^*Pic(B)_{\mathbf{Q}}\subset CH^1(\mathscr{A})_2, \tag{3.4.2}$$

and the first result is proved.

Using the argument above, we just need to show $\pi^*Pic(B)_{\mathbf{Q}}\supset CH^1(\mathscr{A})_2$ to get the second result. The decomposition of $CH^1(\mathscr{A})_{\mathbf{Q}}$ given above shows that $CH^1(\mathscr{A})_2\subset NS_2\oplus\pi^*Pic^0(B)_{\mathbf{Q}}$. We already observed that $NS_2=\pi^*H^2(B,\mathbf{Q})\subset\pi^*Pic(B)_{\mathbf{Q}}$. Thus, the second result follows. □

4. The first proof

We need the following two lemmas.

4.1. Lemma. *Let $\pi:\mathscr{A}\to B$ be an abelian scheme of relative dimension g over a smooth projective curve B, with zero section e. Let L' be a relatively ample line bundle satisfying $e^*L'\cong\mathcal{O}_B$.*

Then there is a line bundle $M\in Pic(B)$ such that

$$\pi_*L'\cong V\otimes_{\mathbf{C}}M,$$

where V is a vector space. In particular,

$$\dim H^0(B,\pi_*L')=r\dim H^0(B,M)\quad\text{and}\quad c_1(\pi_*L')=r\,c_1(M),$$

where $r:=rk(\pi_*L')$, the rank of π_*L'.

Proof. Note that π_*L' is a vector bundle, because L' is relatively ample. The hypothesis $e^*(L')\cong\mathcal{O}_B$ is the "normalization condition" which implies that the vector bundle π_*L' is an antirepresentation of the group scheme $\mathscr{G}(L')$, defined in [6, Section 6], which generalizes the Heisenberg group. The lemma then follows directly from Proposition 2 in Section 6 of [6]. □

4.2. Lemma. *Let $\pi:\mathscr{A}\to B$ be an abelian scheme of relative dimension g over a smooth complete curve B, with zero section e. Let L' be a relatively ample line bundle with $c_1(L')=c_1(L')_0\in CH^1(\mathscr{A})_0$. Then*

$$deg\,c_1(\pi_*L')=-(rk\pi_*L'/2)\,deg\,c_1(\omega_{\mathscr{A}/B}),$$

where $\omega_{\mathscr{A}/B}=det(e^*(\Omega_{\mathscr{A}/B}))$.

Proof. The proof is obtained from the Grothendieck–Riemann–Roch Theorem (GRR):

$$ch(\pi_! L') = \pi_*(ch(L') \cdot td(\Omega^{\text{dual}}_{\mathscr{A}/B}))$$

by considering k^*L' instead of L'. The computation below is copied from [5, Appendice 2].

Since $\Omega^{\text{dual}}_{\mathscr{A}/B} = \pi^* e^*(\Omega^{\text{dual}}_{\mathscr{A}/B})$, where $e^*(\Omega^{\text{dual}}_{\mathscr{A}/B})$ is the Lie algebra bundle associated to $\mathscr{A} \to B$, we can rewrite GRR:

$$ch(\pi_! L') = \pi_*(ch(L')) \cdot td(e^*(\Omega^{\text{dual}}_{\mathscr{A}/B})).$$

Since L' is relatively ample, we have $\pi_! L' = \pi_* L'$. We only consider the codimension 0 and 1 parts of GRR:

$$rk\pi_* L' = \pi_*(c_1(L')^g)/g!,$$

$$c_1(\pi_* L') = \pi_*(c_1(L')^g)/g! \cdot c_1(\omega^{\text{dual}}_{\mathscr{A}/B}))/2 + \pi_*(c_1(L')^{g+1})/(g+1)!.$$

Substituting the first formula in the second gives

$$c_1(\pi_* L') + (rk\pi_* L')c_1(\omega_{\mathscr{A}/B}))/2 = \pi_*(c_1(L')^{g+1})/(g+1)!. \tag{4.2.1}$$

We will prove that both sides of (4.2.1) must be zero by showing that, upon replacing L' by k^*L', the l.h.s. of (4.2.1) is multiplied by k^{2g} but the r.h.s. by k^{2g+2}.

For the l.h.s. note that also k^*L' is relatively ample with $rk\pi_* k^* L' = k^{2g}(rk\pi_* L')$ and, since $\pi_*\ k_* = \pi_*$ and $k_*\ k^* = k^{2g} id$,

$$c_1(\pi_* k^* L') = c_1(\pi_* k_* k^* L') = k^{2g} c_1(\pi_* L').$$

Hence

$$(rk\pi_* k^* L')c_1(\omega_{\mathscr{A}/B})/2 + c_1(\pi_* k^* L') = k^{2g}(rk\pi_* L')c_1(\omega_{\mathscr{A}/B})/2 + c_1(\pi_* L')).$$

Let us look now at the r.h.s. The hypothesis $c_1(L') = c_1(L')_0$ is equivalent to $c_1(k^*L') = k^2 c_1(L')$ so

$$\pi_*(c_1(k^*L')^{g+1})/(g+1)! = k^{2g+2}\pi_*(c_1(L')^{g+1})/(g+1)!$$

as required. □

4.3. Theorem. *Let $\pi : \mathscr{A} \to B$ be an abelian scheme of relative dimension g over a smooth projective curve B. Let e be the zero section and let L be a relatively ample line bundle on $\mathscr{A}$. If*

$$deg(c_1(L) \cdot e(B)) < \tfrac{1}{4} deg\, c_1(\omega_{\mathscr{A}/B}), \tag{4.3.1}$$

then L is not effective.

Proof. Let $c_1(L)=c_1(L)_0+c_1(L)_1+c_1(L)_2$ be the decomposition of $c_1(L)$ in $CH^1(\mathscr{A})_{\mathbf{Q}}$ with $k^*c_1(L)_i=k^{2-i}c_1(L)_i$ (so in particular $(-1)^*c_1(L)_i=(-1)^{2-i}c_1(L)_i$). We define the line bundle

$$L_S:=L\otimes(-1)^*L.$$

The decomposition of $c_1(L_S)$ is

$$c_1(L_S)_0=2c_1(L_S)_0, \qquad c_1(L_S)_1=0, \qquad c_1(L_S)_2=2c_1(L)_2.$$

Thus, by Remark 3.3, it follows from the hypothesis (4.3.1) that

$$deg(c_1(L_S)\cdot e(B))=deg(2c_1(L)_2\cdot e(B))<\tfrac{1}{2}deg\, c_1(\omega_{\mathscr{A}/B}). \tag{4.3.2}$$

On the other hand, if L_S is not effective, i.e. $H^0(\mathscr{A},L_S)=0$, then $H^0(\mathscr{A},L)=0$, i.e. L is not effective, so we just need to show that L_S is not effective.

We now consider the line bundle L' on $\mathscr{A}$ defined by

$$L':=L_S\otimes\pi^*e^*L_S^{-1}.$$

It satisfies $e^*L'\cong\mathcal{O}_B$. Moreover, since L is relatively ample, so are L_S and L'. We show that $c_1(L')_1=c_1(L')_2=0$.

Note that $c_1(\pi^*e^*L_S^{-1})\in\pi^*Pic(B)_{\mathbf{Q}}=CH^1(\mathscr{A})_2$ (by Lemma 3.4), thus $c_1(L')_i=c_1(L_S)_i$ for $i=0,1$. Therefore also $c_1(L')_1=0$. Moreover, writing $c_1(L_S)_2=\pi^*D$ with $D\in Pic(B)$ we have, using Remark 3.3 and the projection formula

$$e^*L_S=\pi_*(c_1(L_S)\cdot e(B))=\pi_*(c_1(L_S)_2\cdot e(B))=D.$$

Thus, $\pi^*(e^*L_S)=c_1(L_S)_2$ and so $c_1(L')_2=0$.

We have to show $H^0(\mathscr{A},L_S)=0$. Using $L_S=L'\otimes\pi^*e^*L_S$ we have

$$H^0(\mathscr{A},L_S)=H^0(B,\pi_*(L'\otimes\pi^*e^*L_S))=H^0(B,(\pi_*L')\otimes e^*L_S).$$

Applying Lemma 4.1 to L' we get $\pi_*L'\cong V\otimes_{\mathbf{C}}M$ and

$$H^0(\mathscr{A},L_S)=V\otimes H^0(B,M\otimes e^*L_S) \quad \text{with } c_1(\pi_*L')=r\cdot c_1(M).$$

Using $deg(e^*L_S)=deg(c_1(L_S)\cdot e(B))$ and

$$deg(c_1(\pi_*L'))=-(r/2)\,deg(c_1(\omega_{\mathscr{A}/B}))$$

(Lemma 4.2), we find that the degree of the line bundle $M\otimes e^*L_S$ on B is

$$deg(M\otimes e^*L_S)=-\tfrac{1}{2}deg(c_1(\omega_{\mathscr{A}/B}))+deg(c_1(L_S)\cdot e(B)),$$

hence $H^0(\mathscr{A},L_S)=0$ if $deg(c_1(L_S)\cdot e(B))<\frac{1}{2}deg(c_1(\omega_{\mathscr{A}/B}))$. □

4.4. Corollary. *Let $\pi:\mathscr{A}\to B$ be a non-isotrivial family of abelian varieties of relative dimension g, over a smooth projective curve B, and L a relatively ample line bundle.*

If L is effective then for any curve C in $\mathscr{A}$

$$deg(c_1(L)\cdot C)>0.$$

Proof. If $C\subset A_b$, a fiber of π, the statement is true because L_b is ample. If not, π restricts to a surjective map $C\to B$. Let $\tilde{C}$ be the desingularization of C and let $\rho:\tilde{C}\to B$ be the map induced by π. Let $\mathscr{A}_1:=\mathscr{A}\times_B\tilde{C}$ be the base change:

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{\tilde{\rho}} & \mathscr{A} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \tilde{C} & \xrightarrow{\rho} & B. \end{array}$$

Then π_1 has a tautological section e. Defining e to be the zero section, $\pi_1:\mathscr{A}_1\to\tilde{C}$ is an abelian scheme. We have

$$deg(c_1(\tilde{\rho}^*L)\cdot e(\tilde{C}))=deg\tilde{\rho}_*(\tilde{\rho}^*c_1(L)\cdot e(\tilde{C}))=deg(c_1(L)\cdot C).$$

Note that if L is relatively ample and effective then so is $\tilde{\rho}^*L$. Now we apply Theorem 4.3 to $\tilde{\rho}^*L$ and obtain

$$deg(c_1(\tilde{\rho}^*L)\cdot e(\tilde{C}))>0,$$

since $deg(c_1(\omega_{\mathscr{A}_1/\tilde{C}}))>0$ because π, and thus π_1, is not isotrivial. □

4.5. Corollary. *Let $\pi:\mathscr{A}\to B$ be a non-isotrivial family of abelian varieties of relative dimension g over a smooth projective curve B. Let $\mathscr{Z}$ be an effective relatively ample divisor on $\mathscr{A}$ and let C be a curve on $\mathscr{A}$.*

Then

$$C\cap\mathscr{Z}\neq\emptyset. \qquad (4.5.1)$$

Proof. The statement is weaker than the one in Corollary 4.4. □

4.6. Corollary. *Let $\pi:\mathscr{A}\to B$ be a non-isotrivial family of abelian varieties of relative dimension g over a smooth projective curve B, whose general fiber is simple. Let $\mathscr{Z}_1$(resp. $\mathscr{Z}_2$) be an effective cycle of codimension p (resp. $g+1-p$) not contained in a fiber. Then*

$$deg(\mathscr{Z}_1\cdot\mathscr{Z}_2)>0.$$

Proof. After a change of base we may assume that π is an abelian scheme (see the proof of Corollary 4.4). The condition $deg(\mathscr{Z}_1\cdot\mathscr{Z}_2)>0$ is equivalent to

$$deg((\mathscr{Z}_1*(-1)^*\mathscr{Z}_2)\cdot e(B))>0,$$

where $*$ is the Pontryagin product.

By construction, the divisor $\mathscr{Z}_1 * (-1)^* \mathscr{Z}_2$ is effective and is not contained in a union of fibers. Under base change the general fiber does not change and effective divisors on simple abelian varieties are ample. By a rigidity argument it follows that $\mathscr{Z}_1 * (-1)^* \mathscr{Z}_2$ is relatively ample. Thus, the result follows from Corollary 4.4. $\square$

5. The second proof

5.1. Let $\pi : \mathscr{A} \to B$ and $\mathscr{X}, C \subset \mathscr{A}$ be as in Corollary 4.5.

If C is contained in a fiber $A_b := \pi^{-1}(b)$ for some $b \in B$ the result follows since $\mathscr{X}$ is relatively ample.

Otherwise, as in the proof of Corollary 4.4, we obtain an abelian scheme (which we again denote by $\pi : \mathscr{A} \to B$) with zero section e and an effective, relatively ample divisor (again denoted by $\mathscr{X}$) and we must show that $e(B) \cap \mathscr{X} \neq \emptyset$.

5.2. We show that we only need to consider the case in which the fiber A_b of π, for $b \in B$ general, is simple.

In fact, if this is not the case, there is a finite map $\rho : B' \to B$ and a diagram

$$\begin{array}{ccc} \mathscr{F} \times_{B'} \mathscr{G} & \xrightarrow{\tilde{\rho}} & \mathscr{A} \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\rho} & B \end{array}$$

where $\mathscr{F} \times_{B'} \mathscr{G}$ is the fiber product of two families of abelian schemes of relative dimensions f and $g - f$ and $\tilde{\rho}$ is surjective. We may assume that $\mathscr{F} \to B'$ is non-isotrivial with simple general fiber.

Let $e_{\mathscr{F}}$ and $e_{\mathscr{G}}$ be the zero sections of $\mathscr{F}$ and $\mathscr{G}$, then $e := (e_{\mathscr{F}}, e_{\mathscr{G}})$ is the zero section of $\mathscr{F} \times_{B'} \mathscr{G}$. We have

$$\mathscr{X} \cap e(B') \quad \text{if and only if} \quad \tilde{\rho}^* \mathscr{X} \cap e'(B') \neq \emptyset.$$

We have the canonical inclusion

$$i_{\mathscr{F}} : \mathscr{F} \xrightarrow{\simeq} \mathscr{F} \times_{B'} e_{\mathscr{G}}(B') \subset \mathscr{F} \times_{B'} \mathscr{G}.$$

Note that $i_F^* \tilde{\rho}^* \mathscr{X}$ is an effective relatively ample divisor on $\mathscr{F}$. Moreover,

$$i_{\mathscr{F}}^* \tilde{\rho}^* \mathscr{X} \cap e_{\mathscr{F}}(B') = i_{\mathscr{F}}^* (\tilde{\rho}^* \mathscr{X} \cap e'(B')) = i_{\mathscr{F}}^* \tilde{\rho}^* (\mathscr{X} \cap e(B')),$$

hence it suffices to show that $i_{\mathscr{F}}^* \tilde{\rho}^* \mathscr{X} \cap e_{\mathscr{F}}(B') \neq \emptyset$.

5.3. If $e(B) \subset \mathscr{X}$, $e(B) \cap \mathscr{X} = e(B) \neq \emptyset$, otherwise $e(B) \cap \mathscr{X}$ is a (finite) set of points. Since the cycles involved are effective, we must prove that

$$deg(\mathscr{X} \cdot e(B)) > 0.$$

We show that our result follows from Proposition 5.4 below. By Remark 2.3.4 we have

$$deg(\mathscr{X} \cdot e(B)) > 0 \quad \text{if and only if} \quad deg(\mathscr{X} \cdot \mathscr{A}[n]) > 0, \tag{5.3.1}$$

where $\mathscr{A}[n]$ is the cycle of n-torsion points.

Note that $\mathscr{A}[n]$ could have several irreducible components and some of them could be contained in $\mathscr{X}$, so the computation of the intersection number can be difficult. However, we may assume that the fiber A_b of π, for $b \in B$ general, is simple (see 5.2). In particular, $\mathscr{X} \cap A_b$ does not contain a translate of an abelian subvariety. A fundamental theorem of Raynaud [7] then shows that the number of the torsion points contained in $\mathscr{X} \cap A_b$ is finite, i.e.

$$card(\mathscr{X} \cap A_b^{\text{tor}}) < \infty.$$

In particular, the number of components of the torsion locus contained in $\mathscr{X}$ is finite.

Now Proposition 5.4 asserts that there are infinitely many n such that $\mathscr{A}[n] \cap \mathscr{X}$ is non-empty. Therefore, there are still infinitely many n for which $\mathscr{A}[n] \cap \mathscr{X}$ is a finite number of points. Since the cycles involved are effective one has $deg(\mathscr{X} \cdot \mathscr{A}[n]) > 0$ for such n, hence $deg(\mathscr{X} \cdot e(B)) > 0$ by (5.3.1).

5.4. Proposition. *Let $\pi : \mathscr{A} \to B$ be a non isotrivial abelian scheme over a smooth projective curve B, with simple general fiber. Let $\mathscr{X}$ be an effective divisor on $\mathscr{A}$.*

Then there exists an infinite subset I of $\mathbf{N}$ such that

$$\mathscr{X} \cap \mathscr{A}[n] \neq \emptyset \quad \textit{for all} \quad n \in I.$$

Proof. By definition

$$\mathscr{A} = R^1\pi_*\mathbf{C}/(\mathscr{F}^1 \oplus R^1\pi_*\mathbf{Z}),$$

where $R^1\pi_*\mathbf{Z}$ is the local system with $(R^1\pi_*\mathbf{Z})_b = H^1(A_b, \mathbf{Z})$ corresponding to the first derived functor, $R^1\pi_*K = R^1\pi_*\mathbf{Z} \otimes K$, for $K = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, the associated K vector bundle and $\mathscr{F}^1$ is the Hodge bundle. Over a point b this is just the definition of abelian variety $A_b = H^1(A_b, \mathbf{C})/(H^{1,0}(A_b) \oplus H^1(A_b, \mathbf{Z}))$.

Moreover, there is a $\mathscr{C}^\infty$ isomorphism

$$\mathscr{A} \simeq R^1\pi_*\mathbf{R}/R^1\pi_*\mathbf{Z}$$

corresponding to the isomorphism $H^1(A_b, \mathbf{R}) \simeq H^1(A_b, \mathbf{C})/H^{1,0}(A_b)$. So we have the diagram

$$\begin{tikzcd} R^1\pi_*\mathbf{R} \arrow[rr, hook] \arrow[dr, "\lambda_R"'] & & R^1\pi_*\mathbf{C} \arrow[dl, "\lambda_C"] \\ & \mathscr{A} & \end{tikzcd} \tag{5.4.1}$$

where λ_R an λ_C are the projections. Note that, by construction, $\lambda_R^* Z = \lambda_C^* Z \cap R^1\pi_*\mathbf{R}$.

Let U be a contractible open neighborhood of a general point $b_0 \in B$. Then over U we have the following trivializations:

$$\Psi_R : R^1\pi_*\mathbf{R} \xrightarrow{\simeq} H^1(A_{b_0},\mathbf{R}) \times U \quad \text{and} \quad \Psi_C : R^1\pi_*\mathbf{C} \xrightarrow{\simeq} H^1(A_{b_0},\mathbf{C}) \times U.$$

We denote by p_1 (any of) the projections onto the first factor. Then we have the maps

$$\Phi_R := p_1\Psi_R : R^1\pi_*\mathbf{R} \to H^1(A_{b_0},\mathbf{R})$$

and

$$\Phi_C := p_1\Psi_C : R^1\pi_*\mathbf{C} \to H^1(A_{b_0},\mathbf{C}).$$

Clearly Ψ_C, Φ_C are holomorphic, while Ψ_R, Φ_R are $\mathscr{C}^\infty$. We denote by $\mathscr{Z}_R$ (resp. $\mathscr{Z}_C$) the restriction to U of the pullback $\lambda_R^*\mathscr{Z}$ (resp. $\lambda_C^*\mathscr{Z}$) and $\mathscr{Z}_{R,b} := \lambda_R^*(\mathscr{Z} \cap A_b)$ and $\mathscr{Z}_{C,b} := \lambda_C^*(\mathscr{Z} \cap A_b)$. By (5.4.1) it follows that

$$\Phi_R(\mathscr{Z}_R) = \Phi_C(\mathscr{Z}_C) \cap H^1(A_{b_0},\mathbf{R}).$$

The proposition follows from the following claim.

Claim. *$\Phi_R(\mathscr{Z}_R)$ contains an open subset of the vector space $H^1(A_{b_0},\mathbf{R})$.*

Indeed, the claim implies that $\Phi_R(\mathscr{Z}_R) \cap H^1(A_{b_0},\mathbf{Q})$ contains an open subset of $H^1(A_{b_0},\mathbf{Q})$. Since $H^1(A_{b_0},\mathbf{Z})$ is discrete in $H^1(A_{b_0},\mathbf{Q})$, this means that $\mathscr{Z}$ intersects an infinite number of torsion points of $\mathscr{A}$ (these torsion points are $(\Phi_R(\mathscr{Z}_R) \cap H^1(A_b,\mathbf{Q}))/H^1(A_b,\mathbf{Z})$).

To prove the claim we note that $\dim_{\mathbf{R}} \mathscr{Z}_R = \dim_{\mathbf{R}} H^1(A_b,\mathbf{R}) = 2g$. Thus, we just need to prove that the differential of $\Phi_{R|\mathscr{Z}_R}$ has maximal rank $2g$ somewhere. First we will check this rank condition for the holomorphic map $\Phi_{C|\mathscr{Z}_C}$ and later we will see that this implies the real case.

We will argue by contradiction. If the rank of $\Phi_{C|\mathscr{Z}_C}$ is no where maximal, then all the fibers of $\Phi_{C|\mathscr{Z}_C}$ have positive dimension. Since $\dim_{\mathbf{C}} \Phi_C(\mathscr{Z}_{C,b}) = 2g - 1$ for all $b \in U$, all the $\mathscr{Z}_{C,b}$ have the same image, i.e.

$$\Psi_C(\mathscr{Z}_C) = \Phi_C(\mathscr{Z}_{C,b_0}) \times U = \mathscr{Z}_{C,b_0} \times U.$$

Since $\mathscr{A}$ is not isotrivial, the vector bundle $\mathscr{F}^1_{|U}$ is not flat. In particular,

$$p_{01}(\Phi_C(\mathscr{F}^1_{|U})) \neq 0,$$

with $p_{01} : H^1(A_{b_0},\mathbf{C}) \to H^{1,0}(A_{b_0})$ the projection, so we can find a vector space

$$W \subset \Phi_C(\mathscr{F}^1_{|U}) \quad \text{with } p_{01}(W) \neq 0.$$

Take now $z_0 \in \mathscr{Z}_{C,b_0}$ and let $\tilde{z} := \Psi_C^{-1}(z_0 \times U)$, a section of $\mathscr{Z}_C$. Since $\lambda_C^{-1}\lambda_C(\tilde{z}(b)) = \tilde{z}(b) + H^{1,0}(A_b) + H^1(A_b,\mathbf{Z})$, the translate

$$\tau_{\tilde{z}}\mathscr{F}^1_{|U} = \{\tilde{z}(b) + H^{1,0}(A_b)\}_{b \in U}$$

is contained in $\mathscr{Z}_C$. So

$$z_0 + W \subset z_0 + \Phi_C(\mathscr{F}^1_{|U}) = \Phi_C(\tau_{\tilde{z}}\mathscr{F}^1_{|U}) \subset \Phi_C(\mathscr{Z}_{C,b_0}) = \mathscr{Z}_{C,b_0}.$$

This means that the analytical subvariety $\mathscr{Z}_{C,b_0}$ of $H^1(A_{b_0}, \mathbf{C})$ contains the affine space, $z_0 + W$, and $p_{01}(z_0 + W_0) \neq 0$.

Then $\lambda_C(z_0 + W) \subset A_{b_0}$ is not a point, so its closure has to be a positive dimensional torus, contained in $\mathscr{Z} \cap A_{b_0}$. But this torus has an infinite number of torsion points. Hence Raynaud's theorem, which we recalled in 5.3, would imply that $\mathscr{Z} \cap A_{b_0}$ contains the translate of an abelian variety, in contradiction with the hypothesis that A_{b_0} is simple.

Now we prove the rank condition for $\Phi_{R|\mathscr{Z}_R}$. We know that the locus where the rank of $\Phi_{C|\mathscr{Z}_C}$ drops is a complex subvariety $Sing(\Phi_{C|\mathscr{Z}_C})$. If the differential of $\Phi_{R|\mathscr{Z}_R}$ has nowhere maximal rank, then $\Phi_R(\mathscr{Z}_R) \subset \Phi_C(Sing(\Phi_{C|\mathscr{Z}_C})) \cap H^1(A_b, \mathbf{R})$. Moreover, the fibers of $\Phi_{R|\mathscr{Z}_R}$ must have positive dimension. Thus, they will be in an union D of irreducible components of $Sing(\Phi_{C|\mathscr{Z}_C})$ where also $\Phi_{C|\mathscr{Z}_C}$ has fibers of positive dimension. Then, in particular, $\dim_{\mathbf{C}} \Phi_C(D) < 2g - 1$. So $\dim_{\mathbf{R}} \Phi_R(\mathscr{Z}_R) < 2g - 1$, because the real part of a complex analytical variety of complex dimension n has real dimension $\leq n$. Then $\dim_{\mathbf{R}} \Phi_R(\mathscr{Z}_R) = \dim_{\mathbf{R}} \Phi_R(\mathscr{Z}_{R,b}) = 2g - 2$, for all $b \in U$. But this implies that the real parts of the images by Φ_C of $\mathscr{Z}_{C,b}$ are the same for all $b \in U$, so these images must be equal and we saw above that this is impossible. □

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